The Apollonius Circles of rank k

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The purpose of this article is to introduce the notion of **Apollonius circle of rank k** and generalize some results on Apollonius circles.

Definition 1. It is called an **internal cevian of rank k** the line AA_k where $A_k \in (BC)$, such that $\frac{BA}{A_kC} = \left(\frac{AB}{AC}\right)^k$ $(k \in \mathbb{R})$.

If A'_k is the harmonic conjugate of the point A_k in relation to B and C, we call the line AA'_k an **outside cevian of rank k**.

Definition 2. We call **Apollonius circle of rank k** with respect to the side BC of ABC triangle the circle which has as diameter the segment line $A_kA'_k$.

Theorem 1. Apollonius circle of rank k is the locus of points M from ABC triangle's plan, satisfying the relation: $\frac{MB}{MC} = \left(\frac{AB}{AC}\right)^k$.

Proof. Let O_{A_k} the center of the Apollonius circle of rank k relative to the side BC of ABC triangle (see Figure 1) and U, V the points of intersection of this circle with the circle circumscribed to the triangle ABC. We denote by D the middle of arc BC, and we extend DA_k to intersect the circle circumscribed in U'. In $BU'C$ triangle, $U'D$ is bisector; it follows that $\frac{BA_k}{A_kC} = \frac{U'B}{U'C} = \left(\frac{AB}{AC}\right)^k$, so U' belongs to the locus. The perpendicular in U' on $U'A_k$ intersects BC on A''_k, which is the foot of the BUC triangle's outer bisector, so the harmonic conjugate

of A_k in relation to B and C, thus $A_k'' = A_k'$. Therefore, U' is on the Apollonius circle of rank k relative to the side BC, hence $U' = U$.

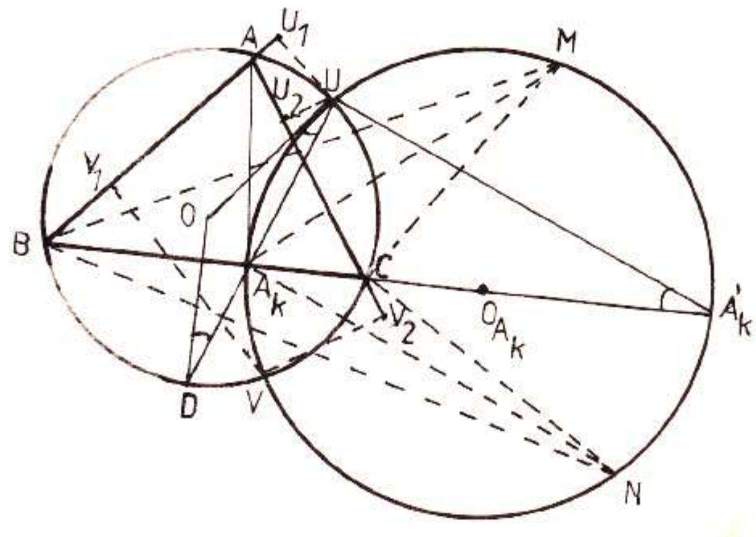

Figure 1

Let M a point that satisfies the relation from the statement; thus $\frac{MB}{MC} = \frac{BA_k}{A_kC}$; it follows – by using the reciprocal of bisector's theorem – that MA_k is the internal bisector of angle BMC. Now let us proceed as before, taking the external bisector; it follows that M belongs to the Apollonius circle of center O_{A_k}. We consider now a point M on this circle, and we construct C' such that $\sphericalangle BNA_k \equiv \sphericalangle A_kNC'$ (thus (NA_k is the internal bisector of the angle $\widehat{BNC'}$). Because $A_k'N \perp NA_k$, it follows that A_k and A_k' are harmonically conjugated with respect to B and C'. On the other hand, the same points are harmonically conjugated with respect to B and C; from here, it follows that $C' = C$, and we have $\frac{NB}{NC} = \frac{BA_k}{A_kC} = \left(\frac{AB}{AC}\right)^k$.

Definition 3. It is called a **complete quadrilateral** the geometric figure obtained from a convex quadrilateral by extending the opposite sides until they intersect. A complete quadrilateral has 6 vertices, 4 sides and 3 diagonals.

Theorem 2. In a complete quadrilateral, the three diagonals' means are collinear (Gauss - 1810).

Proof. Let $ABCDEF$ a given complete quadrilateral (see Figure 2). We denote by H_1, H_2, H_3, H_4 respectively the orthocenters of ABF, ADE, CBE, CDF triangles, and let A_1, B_1, F_1 the feet of the heights of ABF triangle.

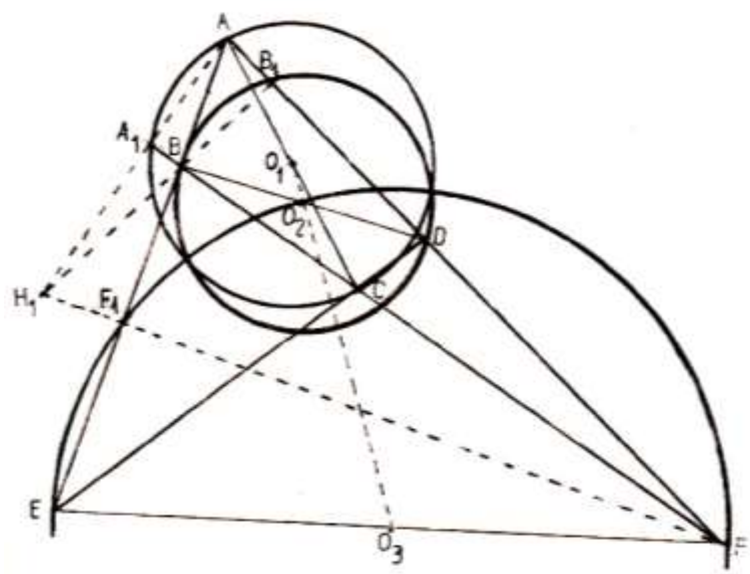

Figure 2

As previously shown, the following relations occur: $H_1A.H_1A_1 - H_1B.H_1B_1 = H_1F.H_1F_1$; they express that the point H_1 has equal powers to the circles of diameters AC, BD, EF, because those circles contain respectively the points A_1, B_1, F_1, and H_1 is an internal point. It is shown analogously that the points H_2, H_3, H_4 have equal powers to the same circles, so those points are situated on the radical axis (common to the circles), therefore the circles are part of a

fascicle, as such their centers – which are the means of the complete quadrilateral's diagonals – are collinear. The line formed by the means of a complete quadrilateral's diagonals is called **Gauss line** or **Gauss-Newton line**.

Theorem 3. The Apollonius circles of rank k of a triangle are part of a fascicle.

Proof. Let AA_k, BB_k, CC_k be concurrent cevians of rank k and AA_k', BB_k', CC_k' be the external cevians of rank k (see Figure 3). The figure $B_k'C_kB_kC_k'A_kA_k'$ is a complete quadrilateral and Theorem 2 is applied.

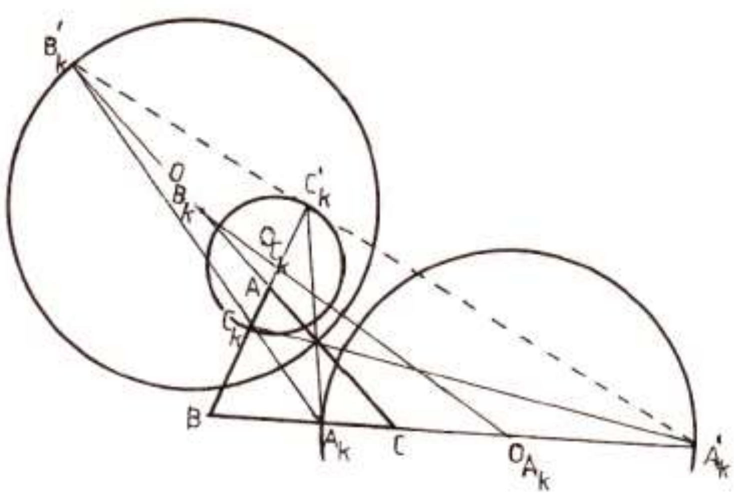

Figure 3

Theorem 4. The Apollonius circles of rank k of a triangle are the orthogonals of the circle circumscribed to the triangle.

Proof. We unite O to D and U (see Figure 1), $OD \perp BC$ and $m(\widehat{A_kUA_k'}) = 90^0$, it follows that $\widehat{UA_k'A_k} = \widehat{ODA_k} = \widehat{OUA_k}$. The congruence $\widehat{UA_k'A_k} \equiv \widehat{OUA_k}$ shows that OU is tangent to the Apollonius circle of center O_{A_k}. Analogously, it can be demonstrated for the other Apollonius circles.

Remark 1. The previous Theorem indicates that the radical axis of Apollonius circles of rank k is the perpendicular taken from O to the line $O_{A_k}O_{B_k}$.

Theorem 5. The centers of Apollonius circles of rank k of a triangle are situated on the trilinear polar associated to the intersection point of the cevians of rank $2k$.

Proof. From the previous Theorem, it results that $OU \perp UO_{A_k}$, so UO_{A_k} is an external cevian of rank 2 for BCU triangle, thus an external symmedian. Henceforth, $\frac{O_{A_k}B}{O_{A_k}C} = \left(\frac{BU}{CU}\right)^2 = \left(\frac{AB}{AC}\right)^{2k}$ (the last equality occurs because U belong to the Apollonius circle of rank k associated to the vertex A).

Theorem 6. The Apollonius circles of rank k of a triangle intersect the circle circumscribed to the triangle in two points that belong to the internal and external cevians of rank $k+1$.

Proof. Let U and V points of intersection of the Apollonius circle of center O_{A_k} with the circle circumscribed to the ABC (see Figure 1). We take from U and V the perpendiculars UU_1, UU_2 and VV_1, VV_2 on AB and AC respectively. The quadrilaterals $ABVC$, $ABCU$ are inscribed, it follows the similarity of triangles BVV_1, CVV_2 and BUU_1, CUU_2, from where we get the relations:

$$\frac{BV}{CV} = \frac{VV_1}{VV_2}, \qquad \frac{UB}{UC} = \frac{UU_1}{UU_2}.$$

But $\frac{BV}{CV} = \left(\frac{AB}{AC}\right)^k$, $\frac{UB}{UC} = \left(\frac{AB}{AC}\right)^k$, $\frac{VV_1}{VV_2} = \left(\frac{AB}{AC}\right)^k$ and $\frac{UU_1}{UU_2} = \left(\frac{AB}{AC}\right)^k$, relations that show that V and U belong respectively to the internal cevian and the external cevian of rank $k+1$.

Definition 4. If the Apollonius circles of rank k associated with a triangle have two common points, then we call these points isodynamic points of rank k (and we denote them W_k, W_k').

Property 1. If W_k, W_k' are isodynamic centers of rank k, then: $W_kA.BC^k = W_kB.AC^k = W_kC.AB^k$; $W_k'A.BC^k = W_k'B.AC^k = W_k'C.AB^k$.

The proof of this property follows immediately from Theorem 1.

Remark 2. The Apollonius circles of rank 1 are the investigated Apollonius circles (the bisectors are cevians of rank 1). If $k = 2$, the internal cevians of rank 2 are the symmedians, and the external cevians of rank 2 are the external symmedians, i.e. the tangents in triangle's vertices to the circumscribed circle. In this case, for the Apollonius circles of rank 2, Theorem 3 becomes:

Theorem 7. The Apollonius circles of rank 2 intersect the circumscribed circle to the triangle in two points belonging respectively to the antibisector's isogonal and to the cevian outside of it.

The proof follows from the Proof of Theorem 6. We mention that the antibisector is isotomic to the bisector, and a cevian of rank 3 is isogonic to the antibisector.

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